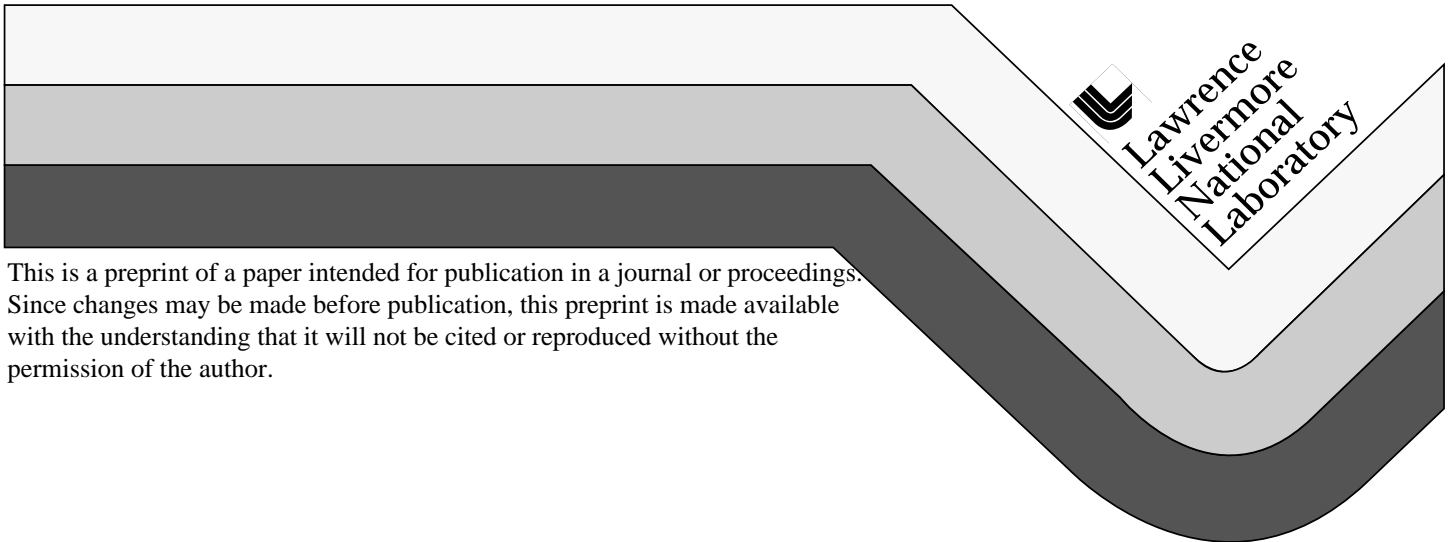


# Exact Solutions of the Wave Equation with Complex Source Locations

Richard W. Ziolkowski

This paper was published in  
Journal of Mathematical Physics  
Volume 26 (4)  
Pages 861-863  
April 1985

June 1984



#### DISCLAIMER

This document was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor the University of California nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial products, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government thereof, and shall not be used for advertising or product endorsement purposes.

1  
2  
3

4  
5  
6

7  
8

EXACT SOLUTIONS OF THE WAVE EQUATION  
WITH COMPLEX SOURCE LOCATIONS

Richard W. Ziolkowski

Lawrence Livermore National Laboratory  
P. O. Box 808, L-156  
Livermore, CA 94550

Submitted to

\_\_\_\_\_

.

.

.

.

.

.

.

.

.

.

.

.

.

.

.

.

### Abstract

New exact solutions of the homogeneous, free-space wave equation are obtained. They originate from complex source points moving at a constant rate parallel to the real axis of propagation; and, therefore, they maintain a Gaussian profile as they propagate. Finite energy pulses can be constructed from these Gaussian pulses by superposition.



A recent article (Brittingham, Ref. 1) has indicated the existence of a new type of solution of the homogeneous, free-space wave equation:

$$\square \phi(\vec{r}, t) = \{\Delta - \partial_{ct}^2\} \phi(\vec{r}, t) = 0 \quad (1)$$

which was termed a Focus Wave Mode (FWM) for its alleged soliton-like properties. It has been found that this FWM is but one of a class of solutions of (1). In particular, assuming the desired direction of propagation is along the z-axis, a solution of the form

$$\phi(\vec{r}, t) = e^{ik(z+ct)} F_k(x, y, z-ct) \quad (2)$$

reduces (1) to a Schrödinger equation; i.e.,

$$e^{-ik(z+ct)} \square e^{ik(z+ct)} F_k = \{\Delta_{\perp} + 4ik \partial_{\tau}\} F_k = 0 \quad (3)$$

where the transverse Laplacian  $\Delta_{\perp} = \partial_x^2 + \partial_y^2$  and the characteristic variables  $(\tau, \sigma) = (z-ct, z+ct)$ . Equation 2 has a symmetric solution ( $\rho^2 = x^2 + y^2$ )

$$F_k(x, y, \tau) = \frac{\exp[-k\rho^2/(z_0 + i\tau)]}{4\pi i(z_0 + i\tau)} \quad (4)$$

that originates at the complex source location  $(\rho, \tau) = (0, iz_0)$ :

$$\{\Delta_{\perp} + 4ik \partial_{\tau}\} F_k(x, y, \tau) = \delta(\rho) \delta(\tau - iz_0) \quad (5)$$

the right-hand side being identically zero for a point in real space-time.

Clearly, the source location  $z = iz_0 + ct$  moves parallel to the real  $z$ -axis. Moreover, defining the complex variance  $V = z_0 + i\tau$  so that

$$\frac{1}{V} = \frac{1}{A} - \frac{i}{R} \quad (6)$$

It is recognized immediately that (4) represents a moving Gaussian beam with beam spread  $A = z_0^2 + (\tau^2/z_0)$ , phase front curvature  $R = \tau + (z_0^2/\tau)$ , and normalized beam waist  $(A/k)^{1/2}$ . Consequently, combining (2) and (4), the fundamental solution

$$\phi(\vec{r}, t) = \phi_k(\rho, \tau, \sigma) = \frac{\exp[ik\sigma - k\rho^2/(z_0 + i\tau)]}{4\pi i(z_0 + i\tau)} \quad (7)$$

as a modulated, moving Gaussian pulse.

A short time history of  $\text{Re}\phi$  with  $f = kc/2\pi = 3.0 \times 10^9$  and  $z_0 = 1.0$  is shown in Fig. 1. The second subplots are contour plots of the 3-D surface plots given in the first subplot. The Gaussian profile of the pulse is apparent. Notice that this profile is maintained during propagation with only local variations. The latter occur primarily near the profile center  $(\rho, \tau) = (0, 0)$ . The variation in the shape of  $\phi$  with  $k$  is illustrated in Fig. 2. The pulse is concentrated near the  $\rho$ -axis for small  $k$  and becomes more concentrated along the  $z$ -axis for large  $k$ . The unusual features of the plots in Fig. 2c such as the jagged peaks and the ragged contours are artifacts of the coarseness of the computational grid. It has also been demonstrated that the pulse amplitude decreases as  $z_0$  increases.

The FWM solution of Maxwell's equations in [1] is readily obtained from (7) with a Hertzian potential formulation. It is the zeroth-order mode in a sequence of multipoles that can be generated in a cylindrical (rectangular)

geometry by applying Laguerre (Hermite) polynomial operators to the fundamental Gaussian mode (7). Dr. Brittingham has brought to my attention that Bélanger (Ref. 2) also recognized this point. However, contrary to the original article [1] and to [2], Fig. 1 demonstrates that the solution is neither focused nor packet-like nor a boost solution (translationally invariant). Moreover, recognizing that these pulses originate from complex source locations connects these results to a large body of literature. In particular, the concept that a Gaussian beam is equivalent paraxially to a spherical wave with a centre at a (stationary) complex location was introduced by Deschamps (Ref. 3) and was later used extensively by Felsen (for example, see Ref. 4) to model the propagation and scattering of Gaussian beams. In contrast to those beam descriptions, (7) is an exact solution of (1). On the other hand, the fundamental Gaussian pulse satisfies all of the properties associated with Gaussian beams. For example, its propagation through an optical system will be described by the ABCD transformation law. [5, Sec.6.7]

The approach that led from (1) to (7) can also be used to define Gaussian pulse solutions of related equations of interest. Consider the Klein-Gordon equation:

$$\square \psi - \mu^2 \psi = 0 \quad (8)$$

where  $\mu = mc/\hbar$ . It has the exact axially symmetric solution

$$\psi(\vec{r}, t) = \exp(-i\mu^2 \tau/4k) \phi(\vec{r}, t) = e^{ik_{\text{eff}} \sigma} F_k(x, y, \tau) \quad (9)$$

where the effective modulation frequency

$$\omega_{\text{eff}} = k_{\text{eff}} c = (k - \mu^2/4k) c = [1 - (mc/2\hbar k)^2] kc \quad (10)$$

has been modified by the mass of the particle. Note that the modulation disappears when  $\hbar k = mc/2$ . Similarly, the wave equation in a transverse quadratic medium

$$\square \psi - (\epsilon_0 + \epsilon_x x^2 + \epsilon_y y^2) \psi = 0 \quad (11a)$$

reduces to a harmonic oscillator Schrödinger equation

$$4ik\partial_\tau F_k = -\Delta_\perp F_k + (\epsilon_0 + \epsilon_x x^2 + \epsilon_y y^2) F_k. \quad (11b)$$

Restricting the problem to one transverse spatial dimension, one can obtain an exact solution from path integral literature (e.g., Ref. 6, Chap. 6) that is readily converted to one originating from a complex source location. A modified Gaussian pulse is obtained. In fact, when the transverse medium coefficients are small, the results reduce to those discussed above. One should then be able to modify standard quantum electronic results (e.g., Ref. 5, Chap 6) to apply directly to these exact complex centre pulse solutions.

A strong objection to the results in [1] has been raised essentially because the solution (7) has infinite energy. This is not a drawback per se. Plane wave solutions of (1) also share the infinite energy property and are commonly employed in constructing physical signals. The Gaussian pulse solutions offer a new set of modes that can be used to construct finite energy solutions of (1). In particular, the function

$$\begin{aligned}
f(\vec{r}, t) &\equiv h(\rho, \tau, \sigma) = \int_0^{\infty} dk F(k) \phi_k(\rho, \tau, \sigma) \\
&= \frac{1}{4\pi i(z_0 + i\tau)} \int_0^{\infty} dk F(k) e^{-ks(\rho, \tau, \sigma)}
\end{aligned} \tag{12}$$

where  $s(\rho, \tau, \sigma) = -i\sigma + \rho^2/(z_0 + i\tau)$  satisfies (1) in real space-time. The wave number has been restricted to non-negative values in this expansion (as well as assuming that  $z_0 > 0$ ) to guarantee the finiteness of the kernel  $\phi_k$ . The resulting Laplace transform expression yields a rich class of possible solutions. An inversion formula corresponding to the Gaussian pulse expansion (12) is

$$F(k) = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau \int_0^{\infty} \rho d\rho \psi_k(\rho, \tau, \sigma) h(\rho, \tau, \sigma) \tag{13}$$

where the kernel

$$\psi_k(\rho, \tau, \sigma) = 8 \pi^{1/2} e^{-(\tau/4kz_0)^2} \phi_k^*(\rho, \tau, \sigma) \tag{14}$$

$\phi_k^*$  being the complex conjugate of  $\phi_k$ . Equivalently, the completeness relation

$$\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau \int_0^{\infty} \rho d\rho \psi_k(\rho, \tau, \sigma) \phi_{k'}^*(\rho, \tau, \sigma) = \delta(k-k') \tag{15}$$

is satisfied by  $\psi_k$  and  $\phi_k$ . The density  $\exp[-(\tau/4kz_0)^2] d\tau$  represents a Gaussian measure over  $\tau$  with real variance  $8(kz_0)^2$ ,  $kz_0$  being the source phase distance, which guarantees the finiteness of the  $\tau$ -integration.

Consider, as an example, the spectrum  $F(k) = \exp(-ak)$ . Equation 12 gives

the pulse

$$f(\vec{r}, t) = \frac{1}{4\pi i(z_0 + i\tau)} \left[ \frac{1}{s(\rho, \tau, \sigma) + a} \right] . \quad (16)$$

Setting  $f_+(\vec{r}, t) = f(\vec{r}, t)$  and  $f_-(\vec{r}, t) = f_+(\vec{r}, -t)$  and  $a = z_0 = \gamma$ , the composite pulse

$$\begin{aligned} \psi(\vec{r}, t) &= f_+(\vec{r}, t) - f_-(\vec{r}, t) \\ &= \frac{1}{2\pi} \frac{\gamma(ct)}{[\rho^2 + (z - ct)(z + ct) + \gamma^2]^2 + 4\gamma^2(ct)^2} . \end{aligned} \quad (17)$$

is a real, exact solution of (1). A time sequence of a pulse (17) with  $\gamma = 1.0$  is given in Fig. 3. The pulse has zero amplitude at  $t = 0$  and its maxima occur at  $\rho = z = 0$  for  $0 < t \leq \gamma/c$  and lie on the sphere  $\rho^2 + z^2 = R^2 = (ct)^2 - \gamma^2$  for  $t > \gamma/c$ . Its amplitude on that sphere  $[8\pi\gamma(ct)]^{-1}$ , decreases essentially as  $R^{-1}$  for  $ct \gg \gamma$ . The likeness of these figures to those describing a pebble dropped in a pond precipitated the name "splash pulse". As the figures illustrate, the support of the splash pulse is localized in space and separates space into two regions of null field for  $t \gg \gamma/c$ , the pulse layer being relatively thin. The apparent spikes in the surface subplot in Fig. 3c again are due to the coarseness of the computational grid employed in the graphics routine.

The interaction of two splash pulses is depicted in Fig. 4. The apparent splash centers are the point  $(\rho, z) = (0, 0)$  and the ring  $(\rho, z) = (7.5, 0)$ . The linear nature of the problem is reflected in the simple superposition in the overlap region and the decoupled propagation of each splash pulse.

Several issues remain outstanding and are currently under investigation. Foremost is the possible launchability of pulses derived from the fundamental Gaussian pulses. The physical connection between resonating structures and Gaussian beams (stationary complex centre descriptions) leads one to speculate that such pulses may be associated with some special type of resonator cavity. In addition to the indicated  $k$ -superposition/transform pair, another class of solutions, those constructed by superposition of the complex source location  $z_0$ , may lead to other physically interesting pulses. Finally, with (2) nonlinear wave equations reduce immediately to the corresponding nonlinear Schrödinger equations. For instance, the cubic wave equation

$$\square \phi - \alpha |\phi|^2 \phi = 0 \quad (18)$$

reduces to the cubic Schrödinger equation

$$4ik\partial_\tau F_k = -\Delta_\perp F_k + \alpha |F_k|^2 F_k \quad (19)$$

At least for one transverse dimension, (19) has known soliton solutions [7, Sec.5.3]. Extensions of these solitons to ones associated with complex source locations may yield other physically interesting wave equation solutions.

The author would like to thank Dr. James N. Brittingham for several interesting discussions on his FWM results. This work was performed by the Lawrence Livermore National Laboratory under the auspices of the U.S. Department of Energy under contract W-7405-ENG-48.

### References

1. J. N. Brittingham, J. Appl. Phys., 54(3), 1179 (1983).
2. P. A. Bélanger, to appear in J. Opt. Soc. Am. (1984).
3. G. A. Deschamps, Electronics Letters, 7, No. 23 (1971).
4. L. B. Felsen, Symposia Mathematica, 18, 39 (1976).
5. A. Yariv, Quantum Electronics, 2nd Ed., John Wiley & Sons, N.Y., 1975.
6. L. S. Schulman, Techniques and Applications of Path Integration, John Wiley & Sons, N.Y., 1981.
7. G. L. Lamb, Jr., Elements of Soliton Theory, John Wiley & Sons, N.Y., 1980.

### Figure Captions

Figure 1. A time sequence of the fundamental solution (7) for  $f = kc/2\pi = 3.0 \times 10^9$  and  $z_0 = 1.0$  demonstrates that it is a modulated moving Gaussian pulse: (a)  $t = 0.0$ , (b)  $t = 4.0 \times 10^{-10}$ , (c)  $t = 8.0 \times 10^{-10}$ .

Figure 2. As  $k$  increases, the Gaussian pulse profile becomes more concentrated along the  $z$ -axis than along the  $\rho$ -axis:  
(a)  $f = 3.0 \times 10^7$ , (b)  $f = 3.0 \times 10^9$ , (c)  $f = 3.0 \times 10^{11}$ .

Figure 3. Time sequence of the splash pulse (17) with  $\gamma = 1.0$ :  
(a)  $t = 8.0 \times 10^{-11}$ , (b)  $t = 2.1 \times 10^{-10}$ , (c)  $t = 8.0 \times 10^{-10}$ .

Figure 4. The interaction of two splash pulses ( $\gamma = 1.0$ ) confirms the linearity of the problem. The splash centers are  $(\rho, z) = (0, 0)$  and  $(\rho, z) = (7.5, 0)$ .



# fundamental gaussian pulse

frequency =  $3.000 \cdot 10^9$  z0 =  $1.000 \cdot 10^9$  time = 0.000

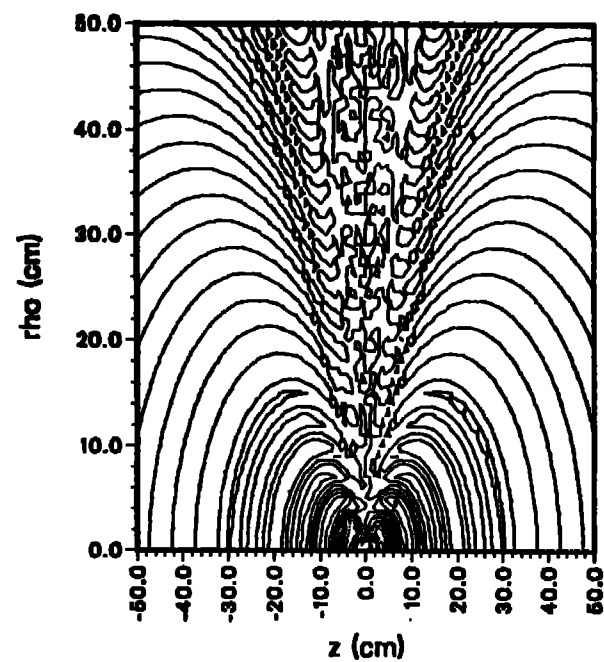
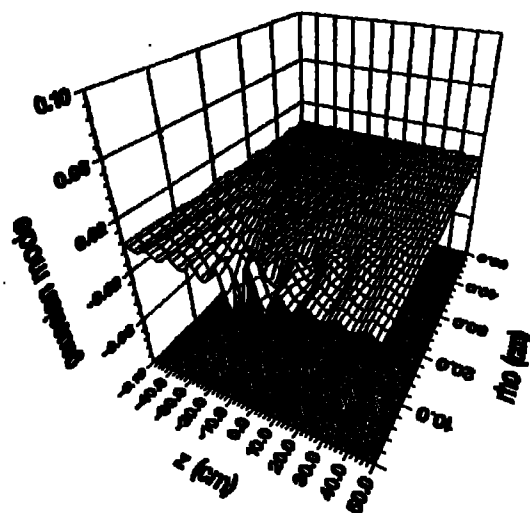


Figure 1a.

# fundamental gaussian pulse

frequency =  $3.000 \cdot 10^9$     $z_0 = 1.000 \cdot 10^9$    time =  $4.000 \cdot 10^{-10}$

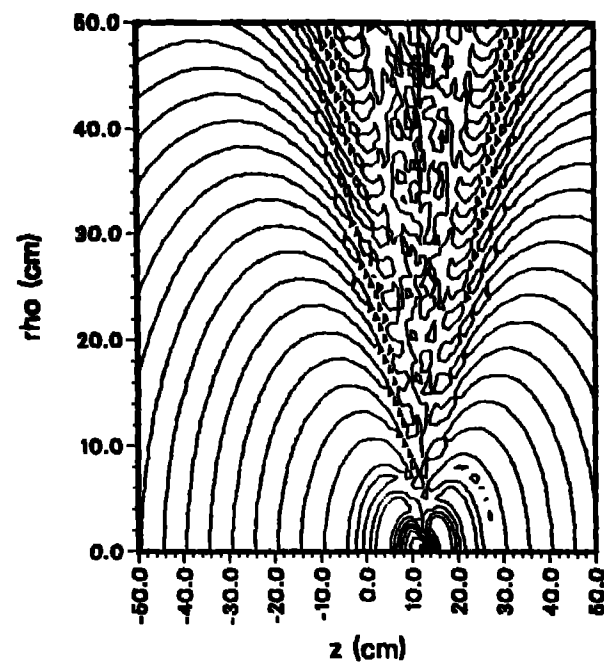
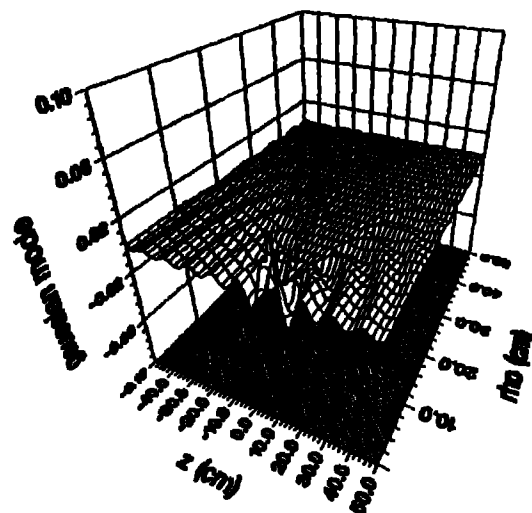


Figure 1b.

# fundamental gaussian pulse

frequency =  $3.000 \cdot 10^9$  z0 =  $1.000 \cdot 10^0$  time =  $8.000 \cdot 10^{-10}$

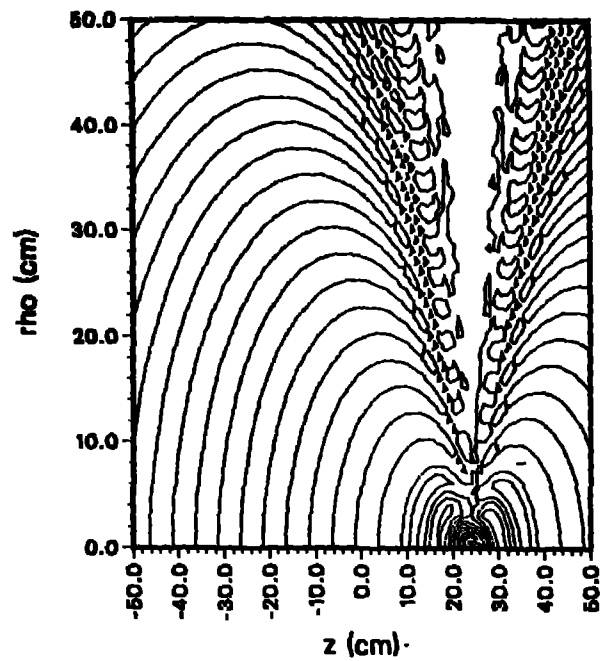
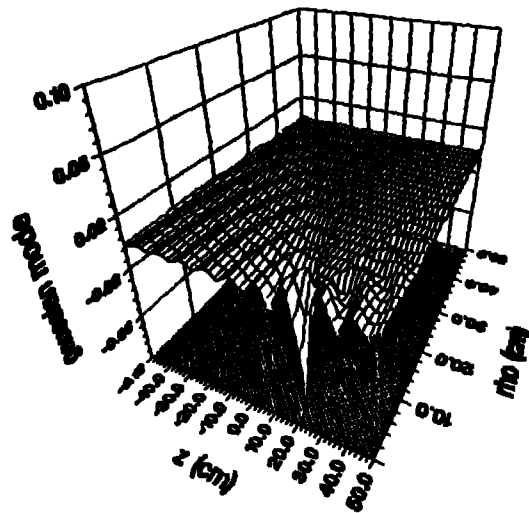


Figure 1c.

# fundamental gaussian pulse

frequency =  $3.000 \times 10^7$  z0 =  $1.000 \times 10^0$  time = 0.000

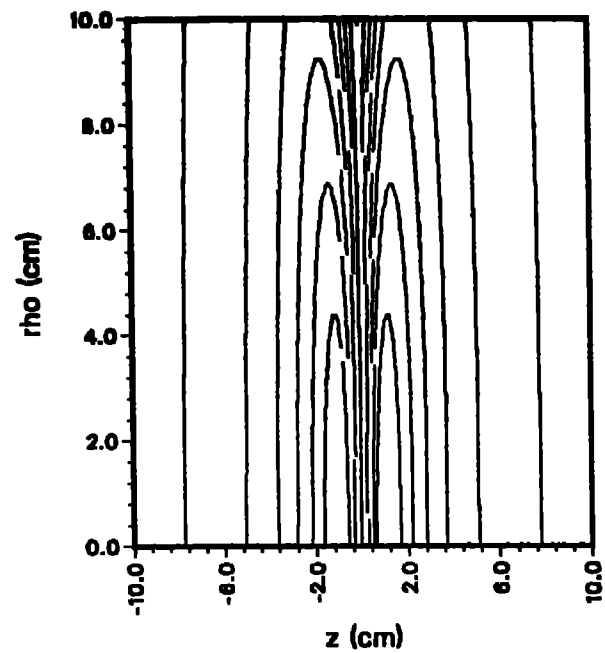
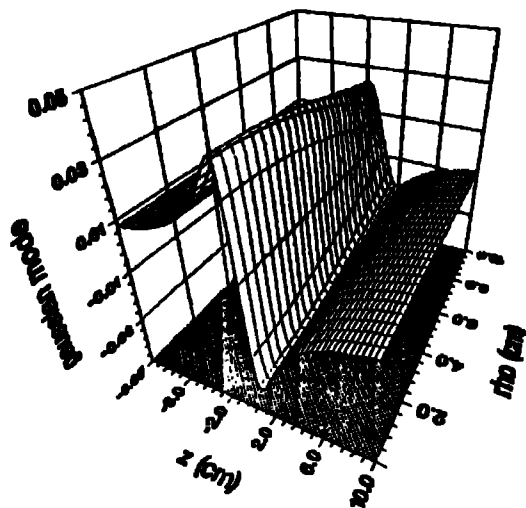


Figure 2a.

# fundamental gaussian pulse

frequency =  $3.000 \times 10^9$  z0 =  $1.000 \times 10^0$  time = 0.000

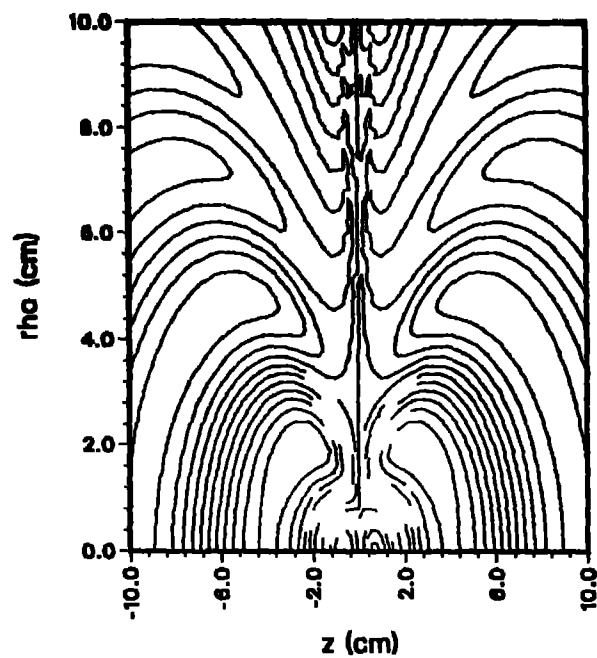
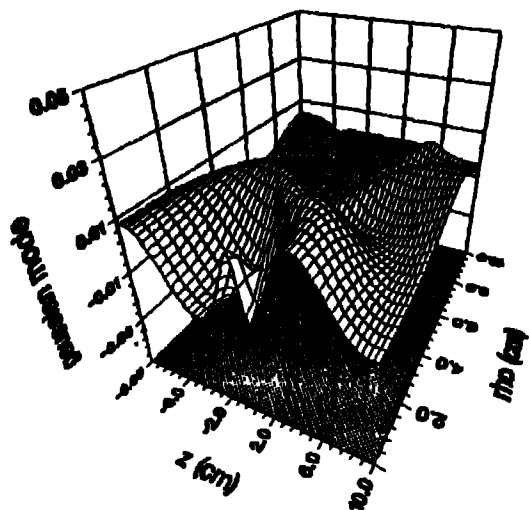


Figure 2b.

# fundamental gaussian pulse

frequency =  $3.000 \cdot 10^{11}$  z0 =  $1.000 \cdot 10^0$  time = 0.000

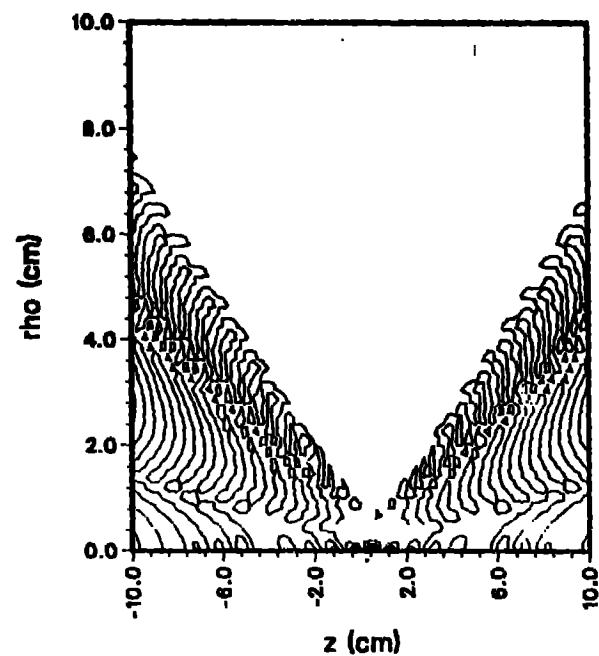
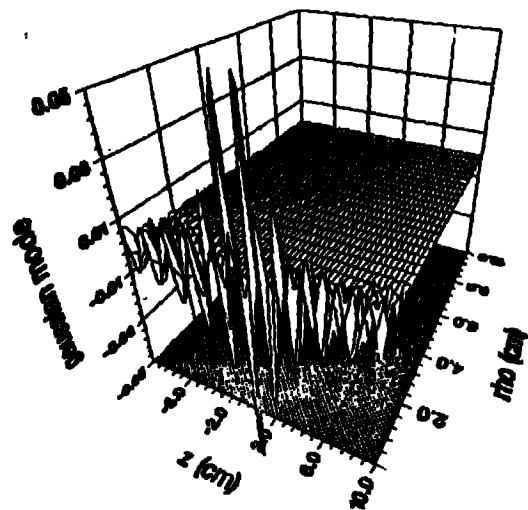
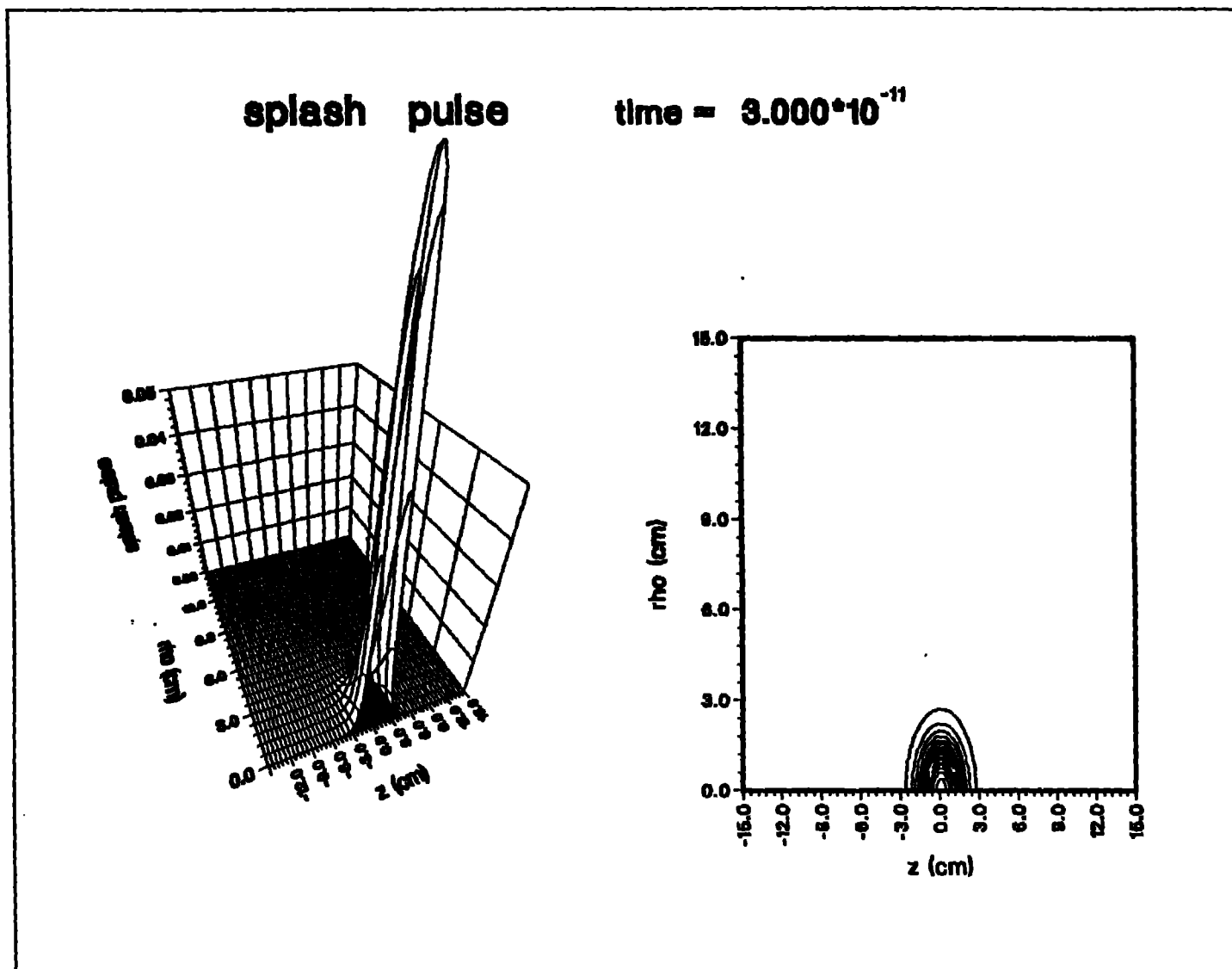


Figure 2c.

Figure 3a.



splash pulse

time =  $2.100 \times 10^{-10}$

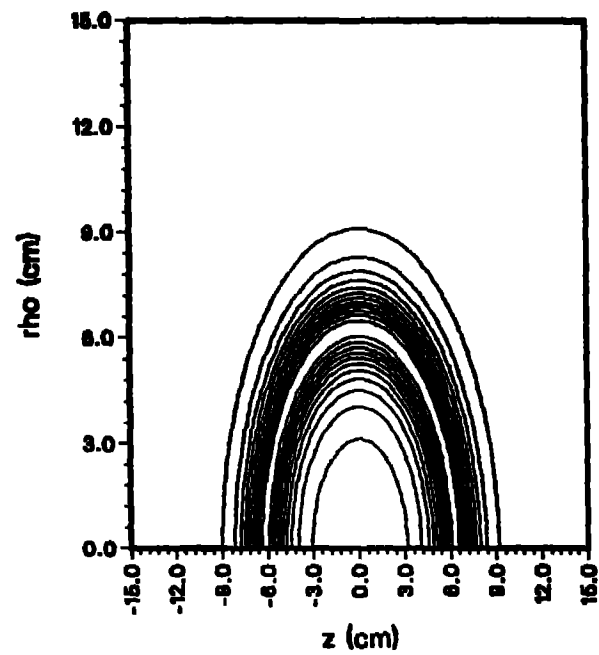
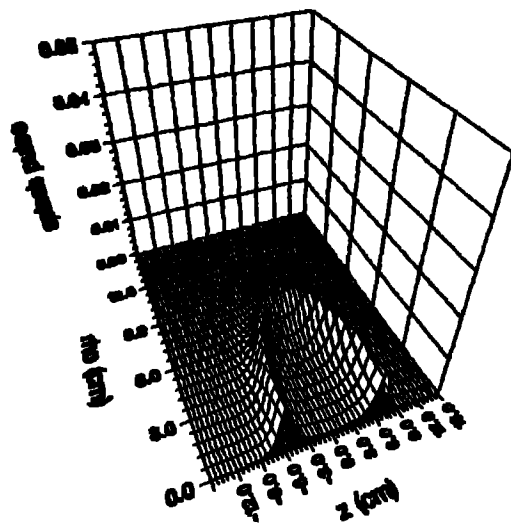


Figure 3b.

splash pulse

time =  $8.000 \times 10^{-10}$

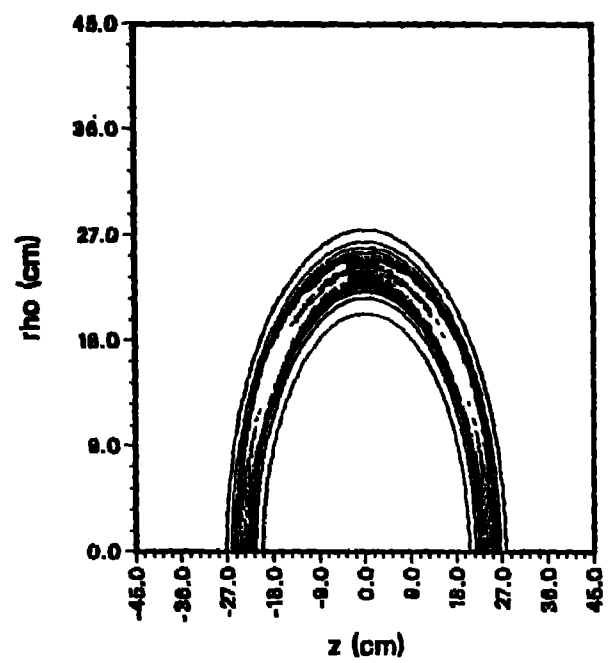
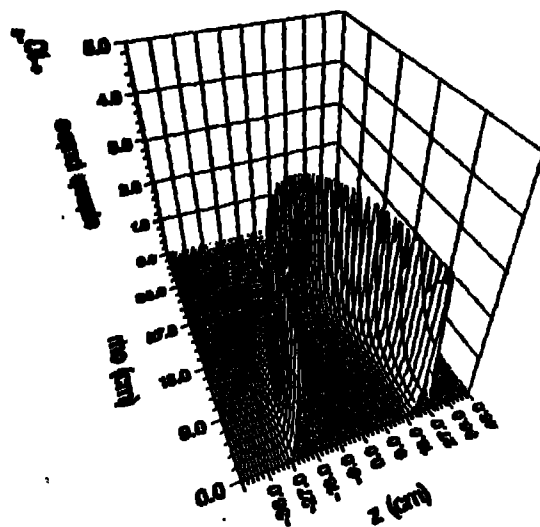


Figure 3c.

# **pulse from an array of splash centers**

**ne = 2   edrho = 7.50   time =  $1.500 \cdot 10^{-10}$**

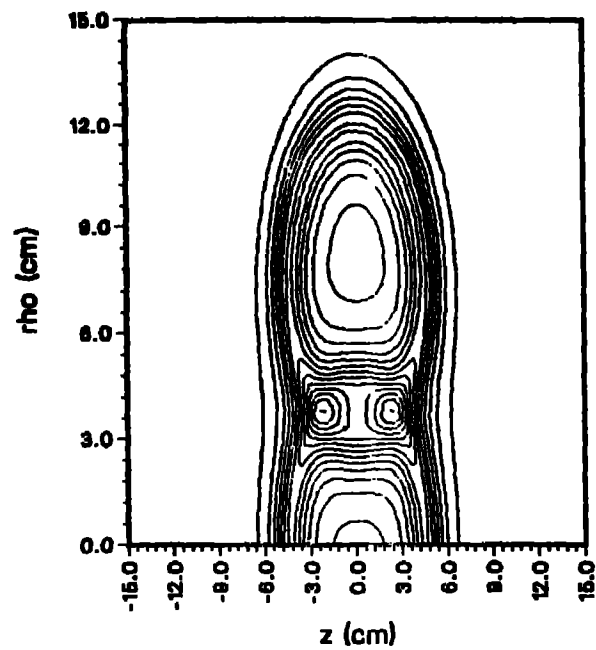
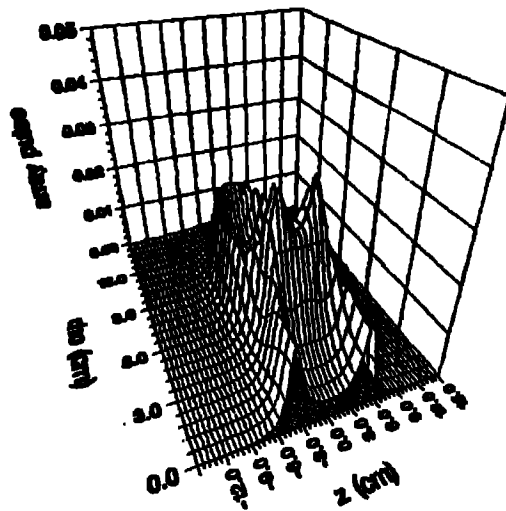


Figure 4.